Kinematics on oblique axes

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Abstract

We solve a difficult problem involving velocity and acceleration components along oblique axes, and propose two problems of central force motion to be solved using oblique axes.

Key Words: Oblique axes, plane motion.

PACS numbers: -1.40.-d, 45.50.- j

1 Introduction

Any vector quantity can be resolved into components according to the parallelogram law, both by rectangular and oblique resolution.
In rectangular coordinates, one component is perpendicular to the other. In oblique coordinates, one component has a projection on the other. We have to take this projection into account when finding velocity and acceleration components along these axes.
Because of this, problems involving oblique axes are very difficult. However, once one realizes that a problem requires oblique axes, the solution is not in general that hard, although the derivation of the kinematics on oblique axes is somewhat disgusting.

2 The velocity components

Consider the motion of a particle in a plane. Suppose that the geometry of the motion is such that the velocity of the particle is more conveniently referred to two oblique axes $O\xi$ and $O\eta$ which make angles ϕ and ψ respectively with a fixed direction Ox in the plane, as shown in Fig. 1. Theses angles may vary arbitrarily with time as the particle moves.

Suppose that at the time t the components of the velocity in the directions $O\xi$ and $O\eta$ are u and v, respectively. The *perpendicular* projections of these components along $O\xi$ and $O\eta$ are, respectively,

$$u + v\cos(\psi - \phi) \qquad \text{and} \qquad v + u\cos(\psi - \phi), \tag{2.1}$$

as shown in the figure for the projection on the $O\xi$ axis only.

At the time $t + \Delta t$ the axes $O\xi$ and $O\eta$ take the positions ξ and η, as shown in Fig. 2, with $\xi O\xi = \Delta\phi$ and $\eta O\eta = \Delta\psi$.

Let the components of the velocity along these axes at this time be $u + \Delta u$ and $v + \Delta v$. The perpendicular projections of these components along the axes $O\xi$ and $O\eta$ are, respectively,

$$(u + \Delta u)\cos\Delta\phi + (v + \Delta v)\cos(\psi - \phi + \Delta\psi)$$

and

$$(v + \Delta v)\cos\Delta\psi + (u + \Delta u)\cos(\psi - \phi - \Delta\phi). \tag{2.2}$$

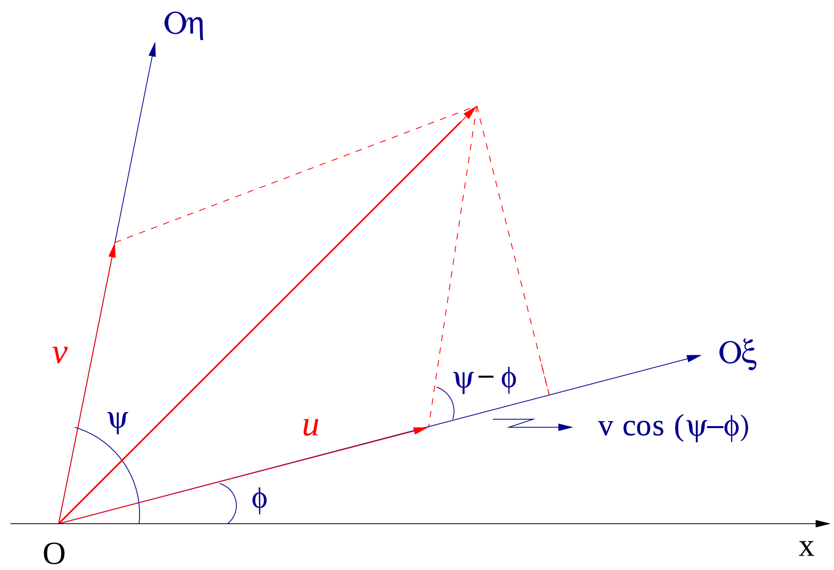

Figure 1: Oblique coordinate system and the components of the velocity along oblique axes. Each component has a projection on the other axis.

By taking the difference between the projections (2.2) and (2.1) of the velocities along the axes $O\xi$ and $O\eta$ at the corresponding times $t + \Delta t$ and t, dividing the result by Δt, and letting Δt go to zero, we obtain the projections of the acceleration along those axes at the time t:

$$\dot{u} + \dot{v}\cos(\psi - \phi) - v\dot{\psi}\sin(\psi - \phi) \tag{2.3}$$

and

$$\dot{v} + \dot{u}\cos(\psi - \phi) + u\dot{\phi}\sin(\psi - \phi), \tag{2.4}$$

where $\dot{u}$ is the limiting value of $\Delta u/\Delta t$, when Δt approaches zero.

3 The acceleration components

Now let a_ξ and a_η represent the components of the acceleration of the particle along $O\xi$ and $O\eta$ at the time t. The *same* relationship (2.1) for velocities hold for accelerations. Thus, the perpendicular projections of the components

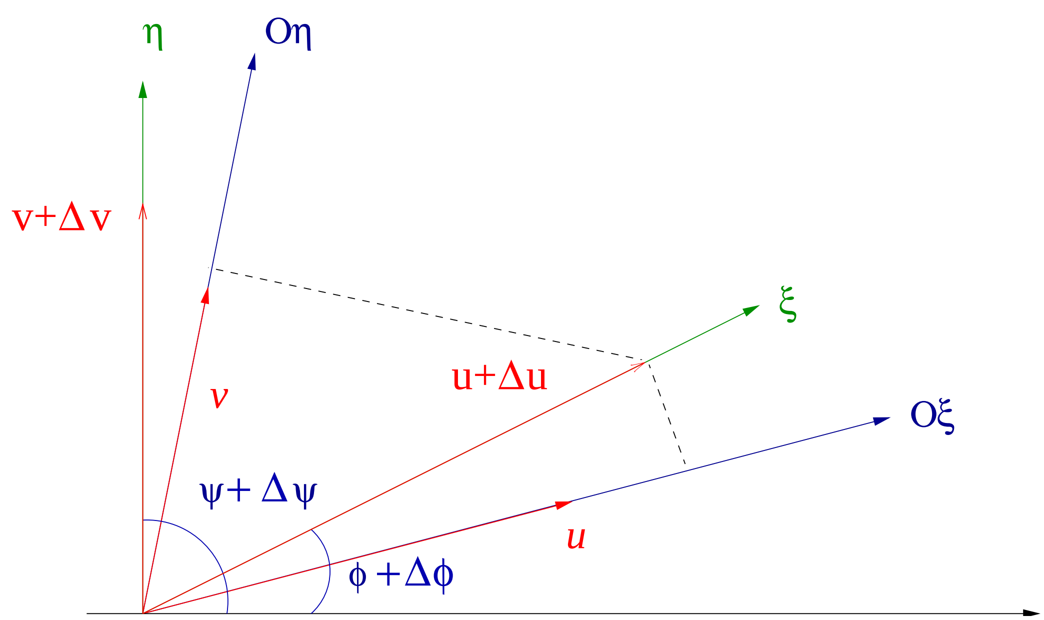

Figure 2: The component of the velocity $u + \Delta u$ along the new axis ξ is projected on the old axis $O\xi$.

along the axes $O\xi$ and $O\eta$ are

$$a_\xi + a_\eta \cos(\psi - \phi) \qquad \text{and} \qquad a_\eta + a_\xi \cos(\psi - \phi). \tag{3.5}$$

On equating (2.2) and (3.5) we obtain

$$a_\xi + a_\eta \cos(\psi - \phi) = \dot{u} + \dot{v} \cos(\psi - \phi) - v\dot{\psi} \sin(\psi - \phi)$$

and

$$a_\eta + a_\xi \cos(\psi - \phi) = \dot{v} + \dot{u} \cos(\psi - \phi) + u\dot{\phi} \sin(\psi - \phi), \tag{3.6}$$

from which we can solve for a_ξ and a_η.

The first two terms in equations (3.6) are the rates of change of the projections (2.1) along fixed axes. The last terms are the consequence of the motion of the axes themselves. (Reference [2] suggests an alternative approach to obtaining these equations.)

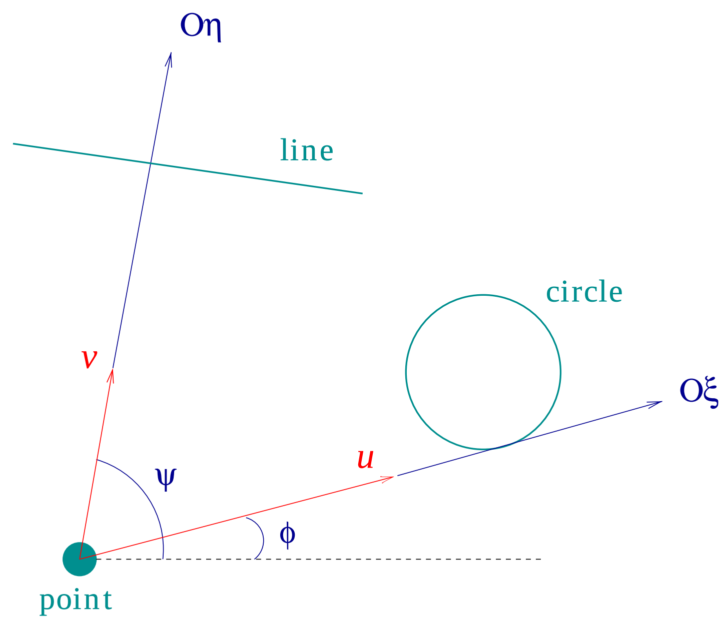

Figure 3: An example of a difficult problem whose solution depends on the kinematics of oblique axes.

4 A difficult problem

As an illustration, consider the following mind boggling problem [3]:

A circle, a straight line, and a point lie in one plane, and the position of the point is determined by the lengths τ of its tangent to the circle and p of its perpendicular to the line. Prove that, if the velocity of the point in made up of components u, v in the directions of these lengths, and if their mutual inclination is θ, the component accelerations will be

$$\dot{u} - \frac{uv}{\tau}\cos\theta, \qquad \dot{v} + \frac{uv}{\tau}.$$

Solution. Take the axis $O\xi$ to be the tangent to the circle, and $O\eta$ to be the axis perpendicular to the line. Set $\theta = \psi - \phi$ and note that ψ does not vary with time. It is then easy to check that equations (3.6), with due change in notation, reduce to the following set of equations

$$a_t + a_p \cos\theta = \dot{u} + \dot{v}\cos\theta$$

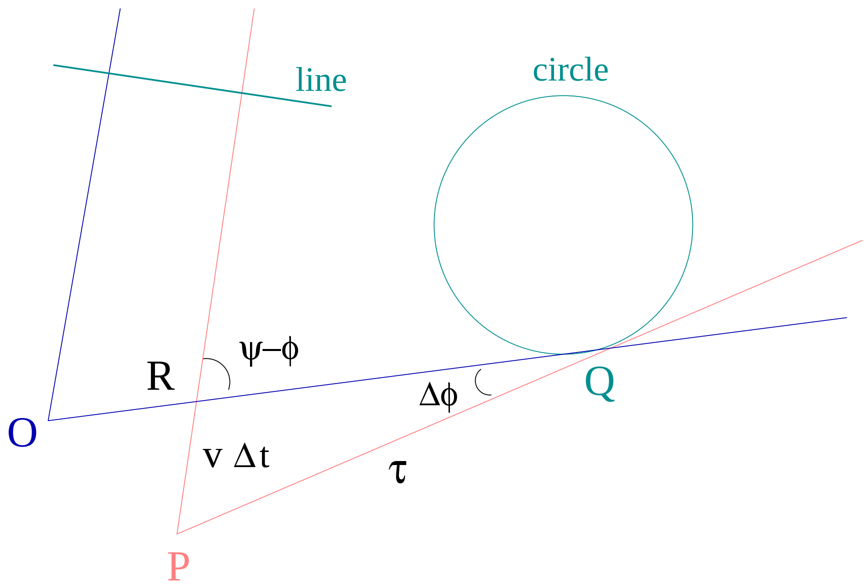

Figure 4: The geometry of the illustrative problem. The particle moves from the point O to the point P. Its position is determined by the tangent line to the circle and also by a perpendicular to a given line in the plane. The tangent lines at two different times meet at Q and make an angle $\Delta\phi$.

and

$$a_p + a_t \cos\theta = \dot{v} + \dot{u}\cos\theta + u\dot{\phi}\sin\theta. \tag{4.7}$$

If we solve (4.7) for a_t and a_p we obtain

$$a_t = \dot{u} - \frac{\cos\theta}{\sin\theta}u\dot{\phi} \qquad \text{and} \qquad a_p = \dot{v} + \frac{u\dot{\phi}}{\sin\theta}. \tag{4.8}$$

To eliminate the variable $\dot{\phi}$ we need to consider the (messy) geometry of the problem. Let the two tangent lines to the circle, drawn from the two positions of the particle at O and P, meet at a point Q, as shown in Fig. 4. Note that the lines OQ and PQ form an angle $\Delta\phi$ with each other at Q. Next, draw from the point P a perpendicular to the given line. Let this perpendicular meet the line OQ at a point R. The perpendicular PR makes an angle $\psi - \phi$ with OQ. For small Δt, P is near O, and we have, approximately, $PQ = \tau$ and $PR = v\Delta t$. The law of sines, applied to the triangle PQR, gives

$$\frac{\Delta\phi}{v\Delta t} = \frac{\sin\theta}{\tau} \qquad \text{or} \qquad \dot{\phi} = \frac{v\sin\theta}{\tau}. \tag{4.9}$$

Substituting $\dot{\phi}$ given above in (4.8), we obtain the desired result.

4.1 Further examples

The equations for velocity and acceleration components on oblique axes can be used to provide solution to problems of motion in a central force field when these problems are phrased as follows.

1. *A particle P moves in a plane in such a way that its velocity has two constant components u and v, with u parallel to a fixed direction in the plane while v is normal to a straight line from the particle to a fixed point O in the plane. Show that the acceleration of the particle is directed along the line OP. (In fact, the particle moves in a ellipse of eccentricity u/v, having O as a focus.)*

2. *A boat crosses a river with velocity of constant magnitude u always aimed toward a point S on the opposite shore directly across its starting position. The rivers also runs with uniform velocity u. Compare this problem with the preceding one. (How far downstream from S does the boat reach the opposite shore?)*

Acknowlgement. I was supported by Fapesp under grant 01/08485-6.

References

[1] R.L. Halfman, *Dynamics*, Addison-Wesley, 1962, p.19

[2] A.S. Ramsey, *Dynamics*, Part II, Cambridge University Press, 1951. (Chapter 3, p.75, example 15.)

[3] E.T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Fourth Edition, Dover Publication, N.Y. 1944. (Chapter 1, p.24, problem 13.)